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Optimal Solutions in an Allocation Process for a Continuum of Traders ⋆

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Abstract. We prove an existence theorem in an allocation process for a continuum of traders in the absence of the convexity assumption on the cost function and under the presence of some economic parameters.

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1. Introduction and Formulation of the Problem

Our concerns in this paper is to study the problem of minimizing the functional J:

$$J(z^1, ..., z^l) = \int_0^1 \int_Q \sum_{k=1}^{l} \lambda_k(t, x) h_k(z^k(t, x)) dx dt \tag{1}$$

among all the integrable functions $z^k : [0, 1] \times Q \to I\!R^n_+$, $k = 1, ..., l$, satisfying the following constraints: for almost all $x \in Q$, every $k = 1, ..., l$;

$$\int_0^t \lambda_k(s, x) \langle z^k(s, x), \nu^k \rangle ds = \int_0^t \lambda_k(s, x) \sum_{j=1}^{n} \nu_j^k z_j^k(s, x) ds \geqslant g^k(t, x), \quad \text{for all } t \in \,]0, 1[\tag{2}$$

$$\int_0^1 \lambda_k(t, x) \nu_j^k z_j^k(t, x) dt = f_j^k(x), \;\; j = 1, ..., n, \tag{3}$$

where, for $k = 1, ..., l$, $z^k = (z_1^k, ..., z_n^k)$; Q is the closed rectangle $\Pi_{i=1}^m [a_i, b_i] \subset I\!R^m$; $g^k : \,]0, 1[\times Q \to \,]0, +\infty[$ is a continuous function; $f_j^k : Q \to \,]0, +\infty[$

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being a positive L^∞-function, $\lambda_k : [0,1] \times Q \to\ ]0,+\infty[$ a continuous function and v^k a non-null vector in $I\!R^n$. We denote by Z the set of functions $z = (z^1, ..., z^l)$ with each $z^k : [0,1] \times Q \to I\!R^n_+$ being an integrable function satisfying (2) and (3).

One of the economic interpretation of our formulation deals with the investigation of markets with a continuum of traders (given by the interval $[0,1]$) in an economy with production. To simplify our interpretation, take $m = 1$. Then, $z^k_j(t,x)$ is the amount of the commodity j to be bought (produced) by the trader t at time x in the place k. Thus, the integral in (1) gives the total expenditure (cost) of the overall coalition under the commodity-assignment $z = (z^1, ..., z^l)$. Each of the positive continuous function λ_k appearing in (1), stands for a subjective discount function associated to the place k, whereas in (2) and (3) $\lambda_k v^k_j$ means a non zero rate of interest applied to the commodity j in the place k. The real-valued function $g^k(t,x)$, which is assumed to be known, is referred as the total commodity bundle required by a $100t$ per cent of the total coalition $[0,1]$ at time x in the place k, whereas the total amount of the single commodity j required by the total coalition at time x in the place k is given by $f^k_j(x)$. Certainly, the following compatibility condition has to be satisfied ($k = 1, ..., l$):

$$\sum_{j=1}^{n} f^k_j(x) \geqslant \limsup_{t\to 1^-} g^k(t,x), \ \text{ a.e. } x \in Q.$$

Under the previous interpretation, inequality (2) and equality (3) have obvious meaning. Then, the problem is to determine an optimal purchase program at minimal cost satisfying the requirements given by (2) and (3).

By recalling that a continuum of traders seems to be more appropiate to describe mathematically the intuitive notion of perfect competition (see [6, 7]), one could think that the appearance of finite places ($k = 1, ..., m$) is in contraposition with this and therefore our model paradoxically would not be well-written. However, our formulation is more general than appear in this context, since it also admits a continuum for the number of places instead of a finite number. This is obtained simply by adding one more component to the variable x and integrate with respect to it. Certainly, this is always possible since there is no restriction for the dimension of x. In any case, our formulation may be considered as a mixed model allowing any finite number of economic parameters (e.g. prices, time, strategies, etc.) varying continuously, and possibly an additional parameter taking values in a finite set.

The main goal of this paper is to prove the existence of optimal solutions to the problem

$$\min_{z\in Z} \int_0^1 \int_Q \sum_{k=1}^{l} \lambda_k(t,x) h_k(z^k(t,x))\,dx\,dt \tag{P}$$

without any convexity assumption on the functions h_k.

If we drop the dependence of the parameter x and the constraint (2), a more general integral than the one considered here has been studied in [17]. In fact, in this case, given f_j^k (being a constant), the problem reduces to prove the exactness of the continuous version of the inf-convolution operator (see [17]), and this is done by imposing a superlinear growth condition on the integrand. The novelty of our formulation lies on the possible dependence of any other parameter x and a constraint of the form (2).

The paper is organized as follows. In Section 2, we recall a result recently proved in [12] within the framework of the Calculus of Variations. Such a result (Lemma 2.1), being optimal in the sense described in Remark 2.2 below, may be considered as one of Liapunov-type but with an additional 'unilateral' condition (see (ii) in Lemma 2.1), which is the novelty. This version is close to the one given in Theorem 2.1 of [2] except for our unilateral condition and extends Lemma 2.2 in [1]. In Section 3, under very mild assumptions on the data, which are discussed as well, we state the theorem ensuring the existence of solutions to problem (P). Its proof together with a possible extension are presented in Section 4.

Related problems can be found in (alphabetical order) [2, 3, 8, 9, 18]. All of them consider maximization problems and therefore the concavity notion is used instead of the convexity, besides a totally different assumption on the growth of the integrands. Problems as those discussed in the preceding papers will be treated elsewhere [13]. A class of problems lacking of classical solutions is studied in [4].

1. A Preliminary Lemma

For the basic definitions about set-valued analysis used here and hereafter, we refer to the book [5, Chapter VIII]. In particular, measurability of set-valued maps $T : Y \subset I\!R^m \to I\!R^n$, is with respect to Lebesgue measure.

In what follows, we use the following notations: $\tilde{Q} \doteq [0, 1] \times Q$, where $Q = \prod_{i=1}^{m}[a_i, b_i] \subset I\!R^m$ and $[a_i, b_i]$ denotes the real interval with end points a_i, b_i, $a_i < b_i$. Given a set $K \subset I\!R^n$, we denote by co K the convex hull of K and by extr K the set of extreme points of K whenever it is convex. The dimension of a convex set K refers to the dimension of the smallest affine space containing K and $\langle\cdot,\cdot\rangle$ stands for the scalar product in $I\!R^n$. The previous notions can be found in [16].

The next result is a slight variant of Lemma 2.3 in [12]. Here we add the function λ. The proof is presented just for the convenience of the reader.

LEMMA 2.1. ([12]) Let $S \subset I\!R^n$ be a k-dimensional relative open simplex with vertices $c_0, c_1, ..., c_k$; $E \subset \tilde{Q} \subset I\!R^{m+1}$ be a measurable set; $v : E \to S$ be a measurable function; $\lambda : \tilde{Q} \to]0, +\infty[$ be a strictly positive continuous function and let $\nu \neq 0$ be a fixed vector in $I\!R^n$. Then, there exists a measurable function $w : E \to$ extr S such that:

(i) $\int_0^1 \lambda(t,x)w(t,x)\chi_E(t,x)dt = \int_0^1 \lambda(t,x)v(t,x)\chi_E(t,x)dt$ for almost all $x \in Q$;

(ii) for every $t \in\ ]0,1[$, we have

$$\int_0^t \lambda(r,x)\langle w(r,x),\nu\rangle\chi_E(r,x)dr \geqslant \int_0^t \lambda(r,x)\langle v(r,x),\nu\rangle\chi_E(r,x)dr \quad \textit{for almost all } x \in Q. \tag{4}$$

Proof. A measurable selection theorem allows us to write $v(t,x) = \sum_0^k p_i(t,x)c_i$ for suitable measurable functions $p_i : E \to [0,1]$ satisfying $\sum_0^k p_i \equiv 1$. Setting $\alpha_i = \langle c_i, \nu\rangle$ for $i = 0, ..., k$, we can assume $\alpha_0 \geqslant \alpha_1 \geqslant ... \geqslant \alpha_k$.

We claim that there exist measurable functions $\delta_i : Q \to [0,1]$, $i = 1, ..., k$, such that $0 \leqslant \delta_i \leqslant \delta_{i+1}$ and, by putting $\delta_0 \equiv 0$, $\delta_{k+1} \equiv 1$, one has

$$\begin{aligned}&\int_{\delta_i(x)}^{\delta_{i+1}(x)} \lambda(t,x)\chi_E(t,x)dt\\ &\quad= \int_0^1 \lambda(t,x)p_i(t,x)\chi_E(t,x)dt \text{ for almost all } x \in Q.\end{aligned} \tag{5}$$

To prove (5), we proceed recursevely as follows. Assuming δ_i is known for $i = 0, ..., j$, we will define δ_{j+1}. Let us consider the function

$$\psi(t,x) = \int_0^t \lambda(r,x)\chi_E(r,x)dr - \int_0^1 \lambda(r,x)\sum_{i=0}^{j} p_i(r,x)\chi_E(r,x)dr.$$

This function is such that: $t \mapsto \psi(t,x)$ is continuous for a.e x; $x \mapsto \psi(t,x)$ is measurable for every t; $\psi(1,x) \geqslant 0$ since $\sum_0^k p_i \equiv 1$ and,

$$\begin{aligned}\psi(\delta_j(x),x) &= \int_0^{\delta_j(x)} \lambda(r,x)\chi_E(r,x)dr - \sum_{i=0}^{j}\int_0^1 \lambda(r,x)p_i(r,x)\chi_E(r,x)dr\\ &= -\int_0^1 \lambda(r,x)p_j(r,x)\chi_E(r,x)dr \leqslant 0.\end{aligned} \tag{6}$$

Thus, by Proposition 2.2 in [12] or Proposition 3.1.2 in [10], the set-valued map $T(x) = \{t \in [\delta_j(x), T] : \psi(t,x) = 0\}$, $x \in Q$, is measurable and then admits at least a measurable selection $\delta_{j+1} : Q \to [0,T]$ (see Theorem 8.1.3 in [5] for instance). In particular, we have $\delta_{j+1}(x) \geqslant \delta_j(x)$ and

$$\begin{aligned}&\int_{\delta_j(x)}^{\delta_{j+1}(x)} \lambda(t,x)\chi_E(t,x)dt\\ &\quad= \int_0^1 \lambda(t,x)p_j(t,x)\chi_E(t,x)dt \text{ for almost all } x \in Q.\end{aligned} \tag{7}$$

This proves claim (5). A desired function satisfying the requirements of the lemma is given by

$$w(t,x) = \sum_{i=0}^{k} c_i \chi_{E_i(x)\cap E}(t,x) = \sum_{i=0}^{k} c_i \chi_{E_i\cap E}(t,x),$$

where $E_i(x) = [\delta_i(x), \delta_{i+1}(x)[\times Q$, for $i = 0, \ldots, k-1$, and $E_k(x) = [\delta_k(x), 1] \times Q$. On the other hand, $E_i = \text{hyp}\,\delta_{i+1} \setminus \text{hyp}\,\delta_i$ for $i = 0, \ldots, k$, here hyp δ_i means the hypograph of the function δ_i defined by $\text{hyp}\,\delta_i \doteq \{(x, y) \in Q \times [0, 1] \ : \ \delta_i(x) \geqslant y\}$.

Let us now prove Part (i).

$$\begin{aligned}
\int_0^1 \lambda(t,x) w(t,x) \chi_E(t,x) dt &= \int_0^1 \lambda \sum_{i=0}^{k} c_i \chi_{E_i(x)\cap E}(t,x) dt \\
&= \sum_{i=0}^{k} c_i \int_{\delta_i(x)}^{\delta_{i+1}(x)} \lambda \chi_E(t,x) d\lambda(t) \\
&= \sum_{i=0}^{k} c_i \int_0^1 \lambda p_i(t,x) \chi_E(t,x) dt \\
&= \int_0^1 \lambda(t,x) v(t,x) \chi_E(t,x) dt. \qquad (8)
\end{aligned}$$

It only remains to prove Part (ii). Fix any $x \in Q$ and t such that $\delta_j(x) \leqslant t \leqslant \delta_{j+1}(x)$ for $j = 0, \ldots, k$. Then

$$\begin{aligned}
&\int_0^t \lambda(r,x) \langle w(r,x), \nu\rangle \chi_E(r,x) dr = \int_0^t \lambda(r,x) \sum_{i=0}^{k} \alpha_i \chi_{E_i(x)\cap E}(r,x) dr \\
&= \sum_{i=0}^{j-1} \alpha_i \int_{\delta_i(x)}^{\delta_{i+1}(x)} \lambda(r,x) \chi_E(r,x) dr + \int_{\delta_j(x)}^{x} \alpha_j \lambda(r,x) \chi_E(r,x) dr \\
&= \sum_{i=0}^{j-1} \alpha_i \int_0^1 \lambda(r,x) p_i(r,x) \chi_E(r,x) dr + \int_{\delta_j(x)}^{t} \alpha_j \lambda(r,x) \chi_E(r,x) dr \\
&= \sum_{i=0}^{j-1} \alpha_i \int_0^t \lambda p_i \chi_E(r,x) dr + \sum_{i=0}^{j-1} \alpha_i \int_t^1 \lambda p_i \chi_E(r,x) dr + \int_{\delta_j(x)}^{t} \alpha_j \lambda \chi_E(r,x) dr \\
&\geqslant \sum_{i=0}^{j-1} \alpha_i \int_0^t \lambda p_i \chi_E(r,x) dr + \alpha_j \int_t^1 (1 - \sum_{i=j}^{k} p_i) \lambda \chi_E(r,x) dr \\
&\quad + \int_{\delta_j(x)}^{t} \alpha_j \lambda \chi_E(r,x) dr
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=0}^{j-1} \alpha_i \int_0^t \lambda p_i \chi_E(r,x) dr + \alpha_j \int_{\delta_j(x)}^1 \lambda \chi_E(r,x) dr - \alpha_j \int_t^1 \lambda \sum_{i=j}^{k} p_i \chi_E(r,x) dr \\
&= \sum_{i=0}^{j-1} \alpha_i \int_0^t \lambda(r,x) p_i(r,x) \chi_E(r,x) dr + \sum_{i=j}^{k} \alpha_j \int_0^t \lambda(r,x) p_i(r,x) \chi_E(r,x) dr \\
&\geqslant \int_0^t \lambda(r,x) \sum_{i=0}^{j-1} \alpha_i p_i(r,x) \chi_E(r,x) dr + \int_0^t \lambda(r,x) \sum_{i=j}^{k} \alpha_i p_i(r,x) \chi_E(r,x) dr \\
&= \int_0^t \lambda(r,x) \sum_{i=0}^{k} \alpha_i p_i(r,x) \chi_E(r,x) dr = \int_0^t \lambda(r,x) \langle v(r,x), \nu \rangle \chi_E(r,x) dr.
\end{aligned} \tag{9}
$$

The latter proves (ii) and the proof of the lemma is concluded. □

REMARK 2.2. ([12]) One can also obtain the existence of another function w satisfying (i) and (ii) with the reverse inequality. This is done by ordering in the opposite sense the α_i, $i = 0, ..., k$ introduced at the beginning of the proof of the lemma. The following is devoted to show that the results of Lemma 2.1 are in some sense optimal. On one hand, one cannot expect that, in addition to (i) and (ii) in Lemma 2.1, also holds an analogue to (i) where the integral is respect to another variable. In other words, assume for simplicity that $n = 1$, $m = 1$, $a_1 = 0$, $b_1 = 1$, $E = \tilde{Q} = [0, 1] \times [0, 1] \subset I\!R^2$, $\lambda = 1$ and $\nu = 1$. Then, one cannot expect that besides satisfying (i) and (ii) in Lemma 2.1, one also has

(iii) $\int_0^1 w(t,x)dx = \int_0^1 v(t,x)dx$ for almost all $t \in [0, 1]$.

On the other hand, the result in the preceding lemma cannot be extended to the case when the set S depends explicitly, at least, on the variable t. In fact, take $n = 2$, $m = 1$, $\lambda = 1$, $E = \tilde{Q} = [0, 1] \times [0, 1]$, $\nu = (0, 1)$, S to be the open interval $\{\ \rho(1,t) :\ \rho \in\ ]0, 1[\ \}$ and $v(t,x) = \frac{1}{2}(1,t)$. Then, it is not difficult to show that there is no function w taking values in $\{(0,0), (1,t)\}$ such that (i) and (ii) continue to be valid. This example was taken from [1].

Finally, it is not hard to realize that, in general, one cannot replace the interval $[0, t]$ in the integral in (ii) for any interval of the form $[t_1, t_2]$ contained in $[0, 1]$.

2. Statement of the Theorem and Discussion of the Assumptions

Given any function $h : I\!R^n \to I\!R \cup \{+\infty\}$, we denote by h^{**} the convexified function of h defined as the greatest convex l.s.c. function not greater than h (see [15]), and by $I\!R^n_+$ the nonnegative orthant $\{\xi \in I\!R^n : \xi_j \geqslant 0,\ j = 1, ..., n\}$. For $z = (z^1, ..., z^l) \in Z$, we use the norm $||z|| = \sum_{k=1}^{l} ||z^k||_{L^1(\tilde{Q}, I\!R^n)}$. We recall that $\tilde{Q} = [0, 1] \times Q$.

We first shall consider the following hypothesis

HYPOTHESIS (H): For every $k = 1, ..., l$, $j = 1, ..., n$, the function $f_j^k : Q \to\ ]0, +\infty[$ is in $L^\infty(Q, I\!R)$, $g^k :\ ]0, 1[\times Q \to\ ]0, +\infty[$ is a continuous function and $h_k : I\!R^n \to [0, +\infty]$ is such that $h_k(\xi) = +\infty$ if and only if $\xi \notin I\!R^n_+$ and

- (h_1) h_k is a lower semicontinuous (and therefore a Borel) function;
- (h_2) there exist: a convex lower semicontinuous monotonic function $\psi : [0, +\infty[\to I\!R^+$ and a constant $\beta \in I\!R$ such that, for all $\xi \in I\!R^n_+$,

$$h_k(\xi) \geqslant \psi(|\xi|) - \beta \text{ where } \lim_{r\to+\infty} \frac{\psi(r)}{r} = +\infty;$$

- (h_3) setting $C^k = \{\xi \in I\!R^n_+ : h_k^{**}(\xi) < h_k(\xi)\}$, we impose that $C^k \subset \bigcup_{i\in I_k} S_i^k$ where I_k is a countable set and each S_i^k is a relative open (bounded) simplex subset of $I\!R^n_+$ such that $h_k^{**} = h_k$ on extr S_i^k and h_k^{**} is affine on every S_i^k. Here $\{S_i^k\}$ are supposed to be disjoint. No assumption if $n = 1$.

Note that we admit as a function h_k, the indicator function of a closed set, i.e., functions of the form $i_C(\xi) = 0$ if $\xi \in C$ and $i_C(\xi) = +\infty$ otherwise for a given closed set C.

We are interested in the following minimization problem

$$\min_{z\in Z} \int_0^1 \int_Q \sum_{k=1}^{l} \lambda_k(t, x) h_k(z^k(t, x))\,dx\,dt \tag{P}$$

where Z is the set of all integrable functions $z = (z^1, ..., z^l)$ satisfying (2) and (3) defined in Section 1.

We now are in a position to state the main result.

THEOREM. *For $k = 1, ..., l$, let v^k be in $I\!R^n$, λ_k be a continuous positive function in $\tilde{Q} = [0, 1] \times Q$ and let f^k, h_k satisfy hypothesis (H) above. If there exists an admisible program $z \in Z$ for which the cost functional J has a finite value and if, in addition, the compatibility condition is satisfied*

$$\sum_{j=1}^{n} f_j^k(x) \geqslant \limsup_{t\to 1^-} g^k(t, x), \text{ a.e. } x \in Q,$$

then problem (P) admits at least a solution in Z.

Before going on, let us discuss briefly the role of each assumption appearing in (H) in the proof of our existence theorem. We start by considering the convexified problem (P^{**}) (same as (P) with h_k^{**} instead of h_k) which will admit a solution by the Direct method of the Calculus of Variations: every minimizing sequence, because of (h_2), has a weakly convergent subsequence in L^1. This fact together with the sequential weak lower semicontinuity of the cost functional associated to (P^{**}), the limit function, say $\tilde{z}$, will be a solution to problem (P^{**}). Now, the goal is to modify each $\tilde{z}^k$ in those points (t, x) for which $z^k(t, x)$ locates in the region where $h_k \neq h_k^{**}$. Assume for the moment that $n = 1$. Then, it is known that (h_2) and the lower semicontinuity of h^k imply that

$$\left\{\xi \in I\!R_+ :\ h_k^{**}(\xi) < h_k(\xi)\right\} = \bigcup_{i \in I} \,]c_i, d_i[, \tag{10}$$

where such intervals (depending on k) are supposed to be disjoint, $-\infty < c_i < d_i < +\infty$, and where i runs over a countable set I depending on k. In addition, one also has $h_k^{**}(c_i) = h_k(c_i)$, $h_k^{**}(d_i) = h_k(d_i)$. We reason for every k and for every $i \in I$ in the following manner. Looking at the points (t, x) for which $\tilde{z}^k(t, x)$ belong to the interval $]c_i, d_i[$, we modify $\tilde{z}^k$ on this set by forcing to take the values in the extreme points of the interval without altering the value of the cost integral. This is done by using Lemma 2.1.

Each interval $]c_i, d_i[$ (an open bounded simplex in $I\!R$) plays the role of the simplex S_i^k in assumption (h_3). Unfortunately, in case $n > 1$ the representation (10) where each interval $]c_i, d_i[$ is substituted by a simplex S_i is not a consequence of (h_2). To see this, assume $l = 1$ and take as a function $\bar{h}$ the minimum of the parabolas $2t^2$ and $t^2 + 1$, and then consider the function $h(\xi) = \bar{h}(|\xi|)$ for $\xi \in I\!R_+^2$. Here $|\xi|$ denotes the Euclidean norm in $I\!R^2$. On the other hand, the advantage of putting an inclusion instead of an equality in (h_3) is exhibited by the next example. Assume again $l = 1$ and take an even function $\bar{h}$ satisfiying, besides (h_2), the following properties: there are $0 < t_1 < t_2$ such that, for $i = 1, 2$, $\bar{h}(t) = \bar{h}(t_i) = \bar{h}^{**}(t_i)$ if $0 \leqslant t \leqslant t_1$; $\bar{h}(t_1) < \bar{h}(t)$ if $t_1 < t < t_2$ and $\bar{h}(t) = \bar{h}^{**}(t)$ for all $t > t_2$ with $\overline{h}^{**\prime}{}_{+}(t_2) > 0$. Now, take the function h given by $h(\xi) = \bar{h}(|\xi|)$, where $|\xi|$ stands for the Euclidean norm. Thus, $h^{**}(\xi) = \bar{h}^{**}(|\xi|)$ and

$$K \doteq \left\{\ \xi \in I\!R_+^n : \bar{h}^{**}(|\xi|) < \bar{h}(|\xi|)\ \right\} = \left\{\xi \in I\!R_+^n :\ t_1 < |\xi| < t_2\right\}.$$

Thus, K is not a countable union of simplices but satisfies the inclusion assumption required in (h_3) as the proposition in next section shows.

3. Proof of the Theorem

(a) We first consider the convexified problem associated to (P):

$$\min_{z \in Z} \int_0^1 \int_Q \sum_{k=1}^{l} \lambda_k(t, x) h_k^{**}(z^k(t, x))\,dx\,dt. \tag{P**}$$

Because of the assumptions on the integrand, the optimal value $\min(P^{**})$ is finite. Take any minimizing sequence (z^p) in Z where $z^p = (z^{1,p}, ..., z^{l,p})$. By the continuity of every λ_k on the compact set $\tilde{Q}$, we can assume that all of them are bounded from below by a positive constant λ. Thus by assumption (h_2) and the de la Vallée-Poussin criterion (see [11, Chapter VII, Theorem 1.3] for instance) the sequence $(z^{k,p})$ is equi-integrable for $k = 1, \ldots, l$. We now apply the Dunford-Pettis compactness criterion (same reference as before) to conclude that there exists a subsequence, still indexed by p, such that $z^{k,p} \rightharpoonup \tilde{z}^k$ in $L^1(\tilde{Q}, \mathbb{R}^n)$. This implies $\tilde{z}^k \geqslant 0$ for $k = 1, \ldots, l$, and, in particular for every $t \in \,]0, 1[$ every Borel set $E \subset Q$, that

$$\int_0^t \int_E \lambda_k(s, y)\langle z^{k,p}(s, y), \nu^k\rangle dy ds \to \int_0^t \int_E \lambda_k(s, y)\langle \tilde{z}^k(s, y), \nu^k\rangle dy ds \quad \text{as } p \to +\infty.$$

Hence, since the sequence $z^p = (z^{1,p}, \ldots, z^{l,p})$ is in Z, one obtains for all $t \in \,]0, 1[$, all Borel set $E \subset Q$:

$$\int_E g^k(t, y)dy \leqslant \int_0^t \int_E \lambda_k(s, y)\langle \tilde{z}^k(s, y), \nu^k\rangle dy ds.$$

Setting $u(t, y) = \int_0^t \lambda_k(s, y)\langle \tilde{z}^k(s, y), \nu^k\rangle dy ds - g^k(t, y)$, the preceding inequality becomes $\int_E u(t, y)dy \geqslant 0$ for all $t \in \,]0, 1[$, all Borel set $E \subset Q$. Thus, for every $t \in \,]0, 1[$ there exists a null set $N(t) \subset Q$ such that $u(t, y) \geqslant 0$ for all $y \in Q \setminus N(t)$. It turns out that $u(t, y) \geqslant 0$ for all $t \in Q_1$, for all $y \in Q \setminus N$ with $N = \bigcup_{t \in Q_1} N(t)$ (independent of t) being a null set, where Q_1 is the set of rational numbers in $]0, 1[$. Since $u(\cdot, y)$ is continuous, the latter implies that $u(t, y) \geqslant 0$ for all $t \in \,]0, 1[$, a.e. $y \in Q$. It follows that for all $t \in \,]0, 1[$ $(k = 1, \ldots, l)$

$$\int_0^t \lambda_k(s, x)\langle \tilde{z}^k(s, x), \nu^k\rangle ds \geqslant g^k(t, x), \quad \text{a.e. } x \in Q.$$

Consequently $\tilde{z} = (\tilde{z}^1, \ldots, \tilde{z}^l)$ satisfies condition (2). Let us prove the second condition (3). Let x be a Lebesgue density point (see [14, Chapter 3] for instance) in the interior of Q for the functions f^k and $\int_0^1 \lambda_k(s, \cdot)\tilde{z}_j^k(s, \cdot)ds$, $k = 1, \ldots, l$, $j = 1, \ldots, n$. The weak convergence of $z^{k,p}$ to $\tilde{z}^k$ in $L^1(\tilde{Q}, \mathbb{R}^n)$, as before, implies that $(k = 1, \ldots, l,\ j = 1, \ldots, n)$

$$\int_0^1 \int_{B(x,\varepsilon)} \lambda_k(s, y) z_j^{k,p}(s, y) dy ds \to \int_0^1 \int_{B(x,\varepsilon)} \lambda_k(s, y)\tilde{z}_j^k(s, y) dy ds \quad \text{as } p \to +\infty.$$

Here, $B(x, \varepsilon)$ is the open ball centred at x and radius ε sufficiently small. By using the Tonelli-Fubini theorem and the facts that z^p is in Z and that x is a density point for the involved functions, we conclude that

$$\int_0^1 \lambda_k(s,x)\nu_j^k \tilde{z}_j^k(s,x)ds = f_j^k(x) \quad \text{for a.e. } x \in Q \ (k = 1, ..., l,\ j = 1, ..., n),$$

since the complement of the set of Lebesgue points has measure zero [14]. Thus, $\tilde{z} = (\tilde{z}^1, ..., \tilde{z}^l)$ is in Z and clearly it is a solution to problem (P^{**}) because of the convexity and lower semicontinuity conditions of h_k^{**} (see [11, Chapter VIII, Theorem 2.1]) and $\lambda_k \geqslant \lambda > 0$.

(b) Let $\tilde{z} \in Z$, $\tilde{z} = (\tilde{z}^1, ..., \tilde{z}^l)$ be any solution to problem (P^{**}). For every $k = 1, ..., l$, we set for every $i \in I_k$, $Q_i^k = \{(t,x) \in \tilde{Q} : \tilde{z}^k(t,x) \in S_i^k\}$, and apply Lemma 2.1 to obtain a measurable function w_i^k taking values in extr S_i^k on Q_i^k, such that, for every $i \in I_k$:

(i) $\int_0^1 \lambda_k(t,x) w_i^k(t,x)\chi_{Q_i^k}(t,x)dt = \int_0^1 \lambda_k(t,x)\tilde{z}^k(t,x)\chi_{Q_i^k}(t,x)dt$ for a.e. x in Q and

(ii) for every $t \in\ ]0,1[$, we have

$$\int_0^t \lambda_k(s,x)\langle w_i^k(s,x), \nu^k\rangle \chi_{Q_i^k}(s,x)ds \\ \geqslant \int_0^t \lambda_k(s,x)\langle \tilde{z}^k(s,x), \nu^k\rangle \chi_{E_i^k}(s,x)ds \quad \text{for a.e. } x \in Q. \tag{11}$$

Put $Q_0^k = \tilde{Q} \setminus \bigcup_i Q_i^k$, and define $z^k : \tilde{Q} \to I\!R^n$ by

$$z^k(t,x) = \tilde{z}^k(t,x)\chi_{Q_0^k}(t,x) + \sum_{i\in I_k} w_i^k(t,x)\chi_{Q_i^k}(t,x).$$

Clearly, this function is measurable and obviously is integrable if I_k is a finite set. In case I_k is numerable, we reason as follows. Taking into account (h_2) and the fact that

$$\int_0^1 \lambda_k(t,x)h_k^{**}(w_i^k(t,x))\chi_{Q_i^k}(t,x)dt = \\ \int_0^1 \lambda_k(t,x)h_k^{**}(\tilde{z}^k(t,x))\chi_{Q_i^k}(t,x)dt \quad \text{for a.e. } x \text{ in } Q, \tag{12}$$

which is a consequence of the affine linearity of h_k^{**} on every S_i^k, we have for every $m \in I\!N$,

$$\lambda \int_{\tilde{Q}} \psi(|\tilde{z}^k|\chi_{Q_0^k} + \sum_{i\leqslant m}|w_i^k|\chi_{Q_i^k}) \leqslant \sum_{i\leqslant m}\int_{Q_i^k} \lambda_k(h_k^{**}(\tilde{z}^k) + \beta) \\ \leqslant \int_{\tilde{Q}} \lambda_k(h_k^{**}(\tilde{z}^k) + \beta), \tag{13}$$

with $\lambda > 0$. Inequality (13) implies that the sequence of functions given by $\tilde{z}^k \chi_{Q_0^k} + \sum_{i \leqslant m} w_i^k \chi_{Q_i^k}$, $m \in \mathbb{N}$ is equi-integrable. Thus, the Vitali convergence theorem [11, Chapter VIII, Corollary 1.3] asserts that z^k is in $L^1(\tilde{Q}, \mathbb{R}^n)$. Moreover, it follows that

$$\int_0^1 \lambda_k(t, x) z^k(t, x) dt = \int_0^1 \lambda_k(t, x) \tilde{z}^k(t, x) dt \quad \text{for a.e. } x \in Q \tag{14}$$

and for every $t \in\ ]0, 1[$

$$\int_0^t \lambda_k(s, x) \langle z^k(s, x), \nu^k \rangle ds \geqslant \int_0^t \lambda_k(s, x) \langle \tilde{z}^k(s, x), \nu^k \rangle ds \quad \text{for a.e. } x \in Q.$$

Thus, $z = (z^1, ..., z^l)$ is in Z. On the other hand, by recalling that h_k^{**} is affine on each S_i^k and since $h_k = h_k^{**}$ on each extr S_i^k, (14) implies

$$\int_0^1 \int_Q \lambda_k(t, x) h_k(z^k(t, x)) dx dt = \int_0^1 \int_Q \lambda_k(t, x) h_k^{**}(\tilde{z}^k(t, x)) dx dt.$$

Hence, $z = (z^1, ..., z^l)$ is a solution to problem (P) since $\tilde{z}$ is a solution to (P^{**}), $\lambda_k \geqslant \lambda > 0$ and $h_k^{**} \geqslant h_k$. Notice that we also conclude that $\min\ (P^{**}) = \min\ (P)$. □

Looking at Part (a) of the proof of our theorem, one concludes that, in case h_k is already a convex function for every $k = 1, ..., l$, the constraint given by (2) may be taken as an equality. We single out this result in the following corollary.

COROLLARY. *In addition to the assumptions of the previous theorem (without (h_3)) assume also that h_k is convex for every $k = 1, ..., l$. Then the problem of minimizing the integral*

$$J(z^1, ..., z^l) = \int_0^1 \int_Q \sum_{k=1}^{l} \lambda_k(t, x) h_k(z^k(t, x)) dx dt,$$

among all the integrable functions $z^k : [0, 1] \times Q \to \mathbb{R}_+^n$, $k = 1, ..., l$, satisfying the following constraints: for almost all $x \in Q$, every $k = 1, ..., l$;

$$\int_0^t \lambda_k(s, x) \langle \nu^k, z^k(s, x) \rangle ds = g^k(t, x), \text{ for all } t \in\ ]0, 1[,$$

$$\int_0^1 \lambda_k(t, x) \nu_j^k z_j^k(t, x) dt = f_j^k(x), \ j = 1, ..., n,$$

admits at least one solution.

REMARK 4.1. By virtue of the first part of Remark 2.2, our theorem admits the variant in which some of the constraints expressed by (2) for $k = 1, ..., l$, may be in opposite sense.

The next proposition, proved in [12], implies that the interior of a k-dimensional convex compact subset C in $\mathbb{R}^n$ is the union of countably many relative open pairwise disjoint simplices (the faces of the k-dimensional simplices in $\mathcal{S}$). This allows us to extend Lemma 2.1 to the case when S is a convex relative open bounded subset of $\mathbb{R}^n$, and therefore also our main theorem by substituting S_i^k, in assumption (h_3), by a convex relative open bounded subset in $\mathbb{R}^n$.

PROPOSITION 4.2. Let C be an n-dimensional compact convex subset of $\mathbb{R}^n$. Then there exists a countable family $\mathcal{S}$ of n-dimensional simplices such that

(i) int $C \subset \cup\{S : S \in \mathcal{S}\}$;
(ii) int $S \cap$ int $S' = \emptyset$ if $S, S' \in \mathcal{S}$ and $S \neq S'$;
(iii) extr $S \subseteq$ extr C for every $S \in \mathcal{S}$.

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References

1. Amar M., and Mariconda C. (1995), A nonconvex variational problem with constraints, SIAM J. Control and Optimization, **33**(1) 299–307.
2. Arkin V.I. & Levin V.L. (1972), Convexity of values vector integrals, theorems on measurable choice and variational problems, Russian Math. Surveys, **27**(3) 21–85.
3. Artstein Zvi, (1974), On a Variational Problem, J. Math. Anal. Appl., **45** 404–415.
4. Artstein Zvi, (1980), Generalized solutions to continuous-time allocation processes, Econometrica, **48**(4) 899–922.
5. Aubin J.P. & Frankowska, H. (1990), *Set-Valued Analysis*, Birkhäuser, Boston, Basel, Berlin
6. Aumann R.J., (1964), Markets with a continuum of traders, Econometrica, 32(1 & 2) 39–50.
7. Aumann R.J., (1966), Existence of competitive equilibria in markets with a continuum of traders, Econometrica, 34(1): 1–17.
8. Aumann R.J. and Perles M., (1965), A Variational Problem Arising in Economics, Journal of Math. Anal. and Appl., 11 488–503.
9. Berliocchi H. and Lasry J.M., (1973), Integrandes Normales et Mesures Parametrees en Calcul des Variations, Bulletin de la Société Mathématique de France, 101: 129–184.
10. Clarke F.H., (1990), *Optimization and Nonsmooth Analysis*, SIAM, Philadelphia.
11. Ekeland I. and Temam R., (1976), *Convex Analysis and Variational Problems*, North-Holland, Amsterdam.

12. Flores-Bazán F. and Perrotta S., (1997), Non-convex variational problems related to a hyperbolic equation, *Technical Report 97-19*, D.I.M., Universidad de Concepción. Accepted for publication in SIAM, J. Control and Optimization.
13. Flores-Bazán F. and Raymond J.-P., A variational problem related to a continuous-time allocation process for a continuum of traders, technical Report 99–18, D.I.M. Universidad de Concepción.
14. Folland G.B., (1984), *Real Analysis: Modern Techniques and Their Applications*, John Wiley & Sons, New York, Toronto, Singapore.
15. Griewank A. and Rabier P.J., (1990), On the smoothness of convex envelopes, Transactions of the Am. Math. Soc., 322(2) 691–709.
16. Rockafellar, R.T., (1972), *Convex Analysis*, Princeton University Press, Princeton, New Jersey.
17. Valadier M., (1970), Integration de convexes fermes notamment d'epigraphes inf-convolution continue, R.I.R.O., **R-2**, 4 année, 57–73.
18. Yaari Menahem E., (1964), On the existence of an optimal plan in a continuous-time allocation process, Econometrica 32(4), 576–590.